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NOTE ON THE GEOMETRICAL TREATMENT OF CONICS.

BY CHARLOTTE ANGAS SCOTT.

1. IT is strange that the treatment of the conic by means of projective flat pencils (or of projective ranges, when the conic is regarded as an envelope) should have so completely displaced the beautiful involution treatment adopted by von Staudt. This enabled him to deal with the points and tangents of the curve simultaneously and symmetrically, with the result that all polar properties are made almost intuitive. In comparison with the simplicity of this, the method of projective pencils is cumbrous; and in comparison with the breadth of von Staudt's treatment, the other is positively niggling. A short account of the development of the theory of conics by von Staudt's method (with a few simplifications) may be of interest.

2. Von Staudt defines involution in a primitive form of the first grade (range of points, flat pencil, or axial pencil) by means of two projective forms on the same base, remarking immediately that this is really a projective pairing of the elements. He might with advantage have employed this, the more characteristic property of the involution, as the defining property; a primitive form of the first grade is arranged in involution if its elements are associated in pairs in such a manner that any figure is projective with its conjugate. This relation holds, for example, if the associated elements are harmonic with reference to a fixed pair of elements. With this definition, we have the usual proofs and properties, namely, that two pairs of elements determine the involution, that the correspondent to any arbitrary element can be determined by a linear construction, that there may be two self-corresponding elements (double elements) but not more, and that these are harmonic with respect to any pair in the involution.

Taking in the next place a primitive form of the second grade, that is, either a plane with all its points and lines, or a point with all the planes and lines through it, it is at once evident that we have to consider the possibility of pairing the elements projectively in two essentially distinct ways; for we may wish to associate elements of the same kind, or elements different in kind. The first does not give any very important results; we consider the second only, and, for definiteness, only as applied to a plane.

3. Supposing such a projective pairing of the elements to have been accomplished, then to a line p corresponds a point P, and *vice versa*; to any figure composed of points $PQR \ldots$ and lines $u\,v\,w \ldots$ there corresponds a figure composed of lines $p\,q\,r \ldots$ and points $UVW \ldots$; to the line AB corresponds the point ab; to collinear points $PQR \ldots$, concurrent lines $p\,q\,r \ldots$, the range and pencil being moreover projective. This relation of a point and line is referred to as polar, the point being the pole of the line, the line the polar of the point; or, we may say, each being polar to the other. The plane thus arranged is a polar system. If the polar of P, namely p, passes through Q, then P lies on q, that is, the polar of Q passes through P; the points P, Q are said to be conjugate, as are also the lines p, q. The point R, common to p, q, has for its polar the line PQ; the triangle PQR has its sides and vertices associated, it is a self-polar or self-conjugate triangle.

Let $A, B, C, \ldots$ be points on any line, which is met by the polars $a, b, c, \ldots$ in $A', B', C', \ldots$; then AA', BB', ... are conjugate. If we start with A', we obtain A, etc; thus the points of the line are paired; and as

$$(AA'BB'CC' \ldots) = (aa'bb'cc' \ldots) = (A'AB'BC'C \ldots),$$

they are projectively paired. Hence pairs of conjugate points on any line form an involution; and pairs of conjugate lines through any point form an involution.

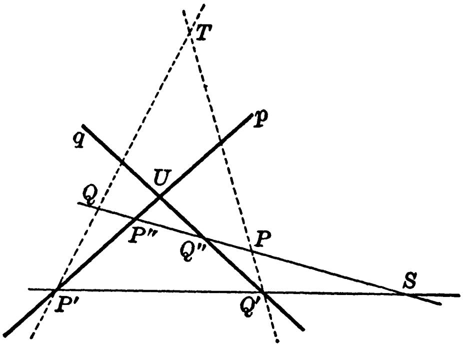

FIG. 1.

4. Let PP', QQ' be any two pairs of conjugates that are not collinear, and let PQ, $P'Q'$ meet in S, PQ', $P'Q$ in T; then S, T are conjugate. (Hesse's theorem.)

For if p, q (fig. 1) meet SPQ in P'', Q'', then PP'', QQ'' are two pairs of conjugate points on the line SPQ, hence they determine the involution of conjugate pairs on this line, and consequently they determine the correspondent to S. This is at once obtained by means of the complete quadrangle $P'Q'TU$; the line TU determines S''. But as S is on PQ, the polar of S goes through $p\,q$, that is, through U. Hence the polar of S is $S''U$, that is, TU; which shows that S and T are conjugate points.

5. This projective pairing of unlike elements of the plane can be accomplished; for a polar system of this nature is determined if we associate the vertices and opposite sides of some one triangle, and then associate any one point of the plane with any one line, these being chosen arbitrarily, and of course not incident on any element of the triangle. To prove this, let ABC be the triangle, Pp the arbitrary pair; let AP and p meet BC in D,D', then D' is associated with AD, and DD' are therefore conjugate points (fig. 2). Thus on the line a two pairs in the involution of conjugate points are known, and the correspondent to any other point is therefore linearly determinable. Hence to construct the line q, associated with any arbitrary point Q, let AQ, BQ, CQ, meet a, b, c in X, Y, Z; then the line q is determined by any two of the conjugates to X, Y, Z, on a, b, c, namely, X', Y', Z'.

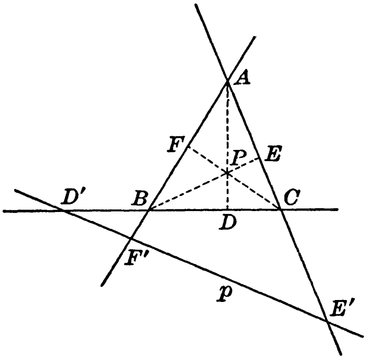

FIG. 2.

That the three points X', Y', Z' are actually collinear, or, what is equivalent, that three collinear points X', Y', Z' satisfy the relations of involution on a, b, c, appears as follows. We have on b, c, the involutions

$$(AC,\ EE',\ YY'),\ (AB,\ FF',\ ZZ').$$

Let BE, CF, meet in P, BY, CZ in Q (fig. 3); then AP, AQ meet a in D, X. Let $E'F'$, $Y'Z'$ intersect in S, and let them meet a in D', X', and the line PQ in P', Q'. Let PQ meet a, b, c in A', B', C', and the lines SA, SB, SC in A'', B'', C''. In virtue of the involution on b,

$$(ACEY) = (CAE'Y');$$

projecting the left hand side of this from B, and the right hand side from S, on to PQ, this gives

$$(C'A'PQ) = (C''A''P'Q').$$

Similarly

$$(B'A'PQ) = (B''A''P'Q').$$

Combining these,

$$(A'B'C'PQ) = (A''B''C''P'Q'),$$

and therefore

$$(B'C'PQ) = (B''C''P'Q').$$

Projecting the left hand side of this from A, and the right hand side from S, on to a, it gives

$$(CBDX) = (BCD'X'),$$

which proves that $(BC,\ DD,\ XX')$ is an involution.

6. Consider the involution of conjugate points on any line; if this has

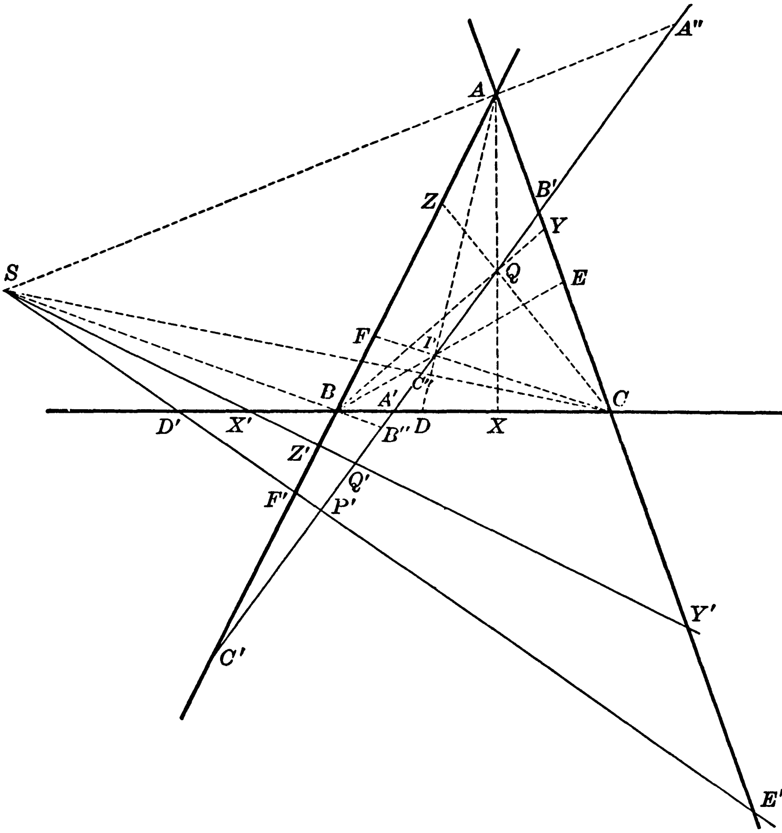

FIG. 3.

double points, each is a point that lies on its own polar, it is conjugate to itself in the plane, and conversely. Similarly all lines that pass through their own poles are the double lines of involution pencils of conjugate lines. Supposing imaginary elements to have been introduced before this stage in the work is reached, a plan which seems highly desirable, the result is that on an arbitrary line there are two self-conjugate points, real or imaginary, and through an arbitrary point there are two self-conjugate lines, real or imaginary.

Let P, p be one such point and line, real or imaginary. First consider any point T on p. The polar of T passes through P, and is distinct from p, hence it does not pass through T. That is to say, no point on p, other than P, lies on its own polar; on the line p there is only the one self-conjugate point P, and to this every point on the line p is conjugate.

In the next place, consider the line t, that is, any line through P. On this there is an involution of conjugate points, which has one double point P, real or imaginary, and consequently another Q, of the same nature. Hence on every line through P, other than p, there is one self-conjugate point distinct from P. There is thus a system of these points, $P\,Q\,R$ . . . ., in perspective with (and consequently in (1, 1) correspondence with) the pencil of lines through any one of them, P. This system is such that any line through one of its points passes through precisely one other, with the exception of one line at every point. These excepted lines, $p\;q\;r$ . . . , form a system in perspective with (and consequently in (1, 1) correspondence with) the range determined on any one of them; the system is such that through any point on one of its lines, p, another line passes, unless the point is P, the pole of p. The point-system $P\,Q\,R$ . . . and the line-system $p\;q\;r$ . . . consequently indicate one and the same curve, which regarded as a locus is of the second order, and regarded as an envelope is of the second class. This curve is called a conic; it has points, P, and tangents, p, meeting the curve at P only.

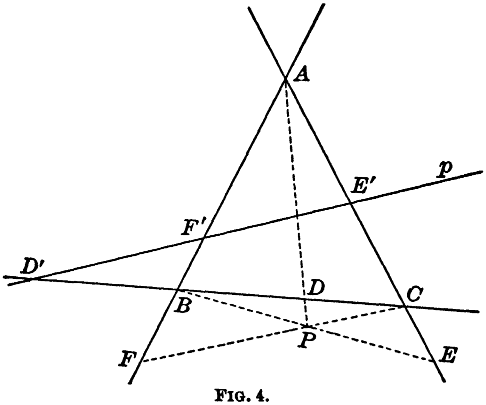

FIG. 4.

7. It may happen that a real polar system gives rise to an imaginary conic (von Staudt, *Beiträge zur Geometrie der Lage*, p. 107). If a point P and its polar p are so placed that it is impossible to pass from P to p without crossing a side of the triangle ABC (or, we may say, if P and p are separated), then the involution of conjugate points on p has imaginary double ele-

ments. The double points on p are determined by the double lines of the involution $P(DD', EE', FF')$; fig. 2 shows one of the two possible arrangements of P, p, and as PE, PE' are necessarily separated by PF, PF', the double lines are imaginary. Similarly if P be outside the triangle, a case which is shown in fig. 4. Now if one point P is separated from its polar p, this is true for every other pair Qq. For referring to fig. 3, XX' belong to the involution determined by BC, DD', hence if D, D' are separated by B, C, this is true of X, X' also, and similarly of Y, Y' and Z, Z'. Thus if P, p are chosen so as to be separated by the triangle ABC, every line meets the conic in imaginary points, hence the conic is entirely imaginary ; while if P, p are not separated by the triangle ABC, two sides of the triangle meet the conic in real points ; the conic is then real, being met by every line through one of these real points in another real point.

The case of an imaginary conic through two or four real points does not present itself in connection with a real polar system ; it arises in the consideration of imaginary polar systems.

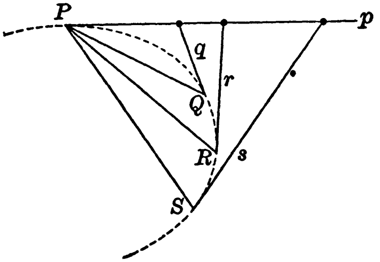

Fig. 5.

8. Since the tangents at P, Q, are the polars of P, Q, they meet in the pole of PQ. Thus the pole of a line is the point of intersection of the tangents at the points where the line cuts the conic ; and the polar of a point is the chord of contact of tangents from the point.

The tangents at Q, R, S, . . meet the tangent at P, that is, p, in the poles of PQ, PR, PS, ... ; hence the projective pairing of the elements of the plane shows that the range determined on p by the tangents at Q, R, S, . . is projective with the pencil determined at P by the points Q, R, S, . . . (fig. 5).

On any line of the plane there is an involution of conjugate points, whose double points are the points where the line meets the conic. Let U, u be a pole and polar, and let any line through U cut the conic in P, Q, and u in U'. Then as U, U', being conjugate points, are harmonic with respect to P, Q, by varying the line through U we obtain the theorem : — the locus of a point harmonically separated from U by the conic is a straight line u, the polar of U. Thus the polar relation of a point and line in the plane system is exactly equivalent to the ordinary polar relation with respect to the conic.

9. It has been shown that the points of a conic subtend at any one point, P, a pencil that is projective with the range determined on p by the tangents. It has now to be shown that the pencils determined at any two points are projective. The proof given by von Staudt is unnecessarily complicated and lengthy; a shorter one is here given, which is really a proof of Chasles' theorem, so simple that it is difficult to believe that it has not been given before.

Let W be the pole of PQ (fig. 6); let R be any point on the conic, and let PR, QR meet QW, PW, in Q', P'; let r meet q, p in U, V. Then since

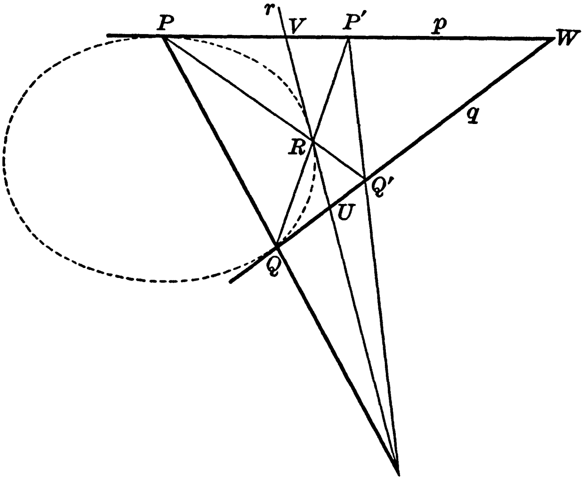

FIG. 6.

Q' lies on the polar of Q and on the polar of V, VQ is the polar of Q'; hence VQ, VQ' are conjugate lines, and are therefore harmonically separated by the tangents from V, that is, by VP, VR. Thus (QQ', UW) is harmonic. Hence since Q, W are fixed points, the ranges described on QW by Q' and U are projective;* that is, symbolically,

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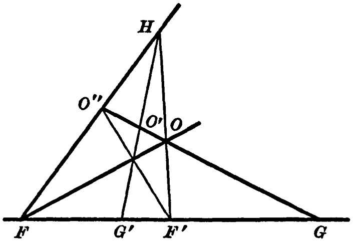

FIG. 7.

LEMMA. If (FF', GG') is harmonic, where F, G are fixed, F', G' variable, then F', G' describe projective ranges. For in the figure by which the harmonic set is determined (fig. 7), we can keep the lines FO, FO'', GO, and the point O', fixed; we have then a series of pencils in perspective,

$$O(F' \ldots) = O(H \ldots) = O'(H \ldots) = O'(G' \ldots);$$

hence

$$\text{range } (F' \ldots) = \text{range } (G' \ldots).$$

$$(Q' \ldots .) = (U \ldots .).$$

Similarly

$$(P' \ldots .) = (V \ldots .).$$

But since V is the pole of PR,

$$\text{range } (V \ldots .) = \text{pencil } P(R \ldots .)$$
$$= \text{range } (Q' \ldots .).$$

Hence the ranges $(U \ldots .)$, $(V \ldots .)$ are projective with one another, and with the pencils $P(R \ldots .)$, $Q(R \ldots)$, which proves not only Steiner's theorem, that the points of a conic are the intersections of corresponding rays of projective pencils, and the tangents to the conic the lines joining corresponding points of projective ranges, but also Chasles' theorem, namely, that four given points on a conic determine at any point of the curve a pencil projective with the range determined on any tangent to the curve by the tangents at the given points.

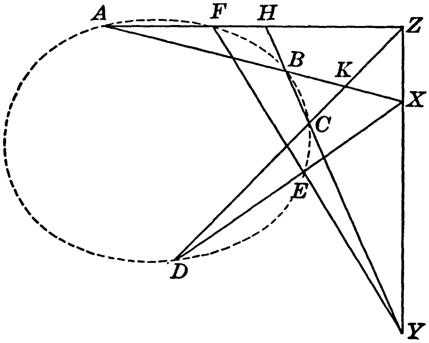

FIG. 8.

10. Von Staudt's own proof of Pascal's theorem (or of Brianchon's theorem) is simple enough. Since the pencils $D(BCEA)$, $F(BCEA)$ are projective (fig. 8), the ranges in which they are cut by any two transversals are projective. Using the transversals BA, BC, this gives

$$(BKXA) = (BCYH),$$

and as these two projective ranges have B in common, they are in perspective. Hence KC, XY, AH are concurrent, that is, CD and AF meet on the line XY.

His proof of Desargues' theorem is unnecessarily long; a simpler one is given by later writers, *e. g.* by Reye.

Let a conic through $ABCD$ cut the transversal XY in P, P'. Then (fig. 9)

$$D(ACPP') = B(ACPP'),$$

therefore

$$(Y'X'PP') = (XYPP') = (YXP'P),$$

which shows that PP' are a pair in the involution XX', YY'. Hence all conics through $ABCD$, including the three line-pairs, cut any transversal in involution.

All the purely projective properties of conics can be readily deduced from these few theorems.

11. A comparison with the treatment of conics based on projective pencils or ranges, as given for example in Reye's *Geometrie der Lage*, or in the article "Geometry" (Section II) in the *Encyclopædia Britannica*, will prove the great superiority of von Staudt's method, which is exhibited specially in everything that relates to poles and polars with reference to the conic.

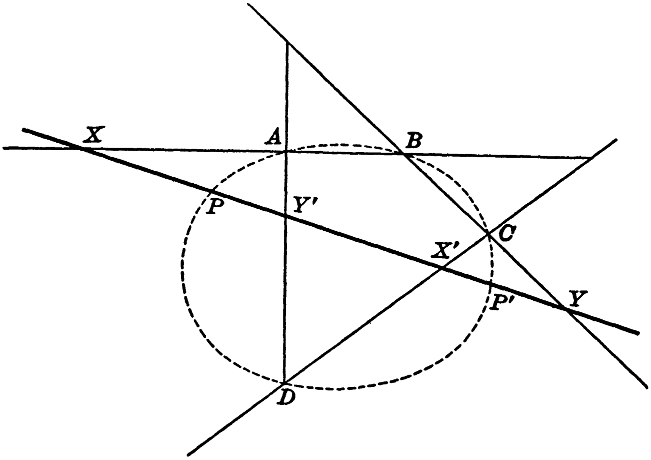

FIG. 9.

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